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Signal detection statistics of stochastic resonators

M. E. Inchiosa* and A. R. Bulsara†
Naval Command, Control and Ocean Surveillance Center, RDT & E Division, Code 524, San Diego, California 92152-5000
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We consider the signal detection performance of networks of coupled overdamped nonlinear dynamic elements driven by a weak sinusoidal signal embedded in Gaussian white noise. In the "stochastic resonance" operating regime, (1) the detection performance exhibits a maximum reflecting the maximum in the signal-to-noise ratio, and (2) coupling significantly enhances the detection performance over that of a single element. Coupling-induced linearization allows the nonlinear system to approach the performance of the linear system which is optimal for our signal detection problem.

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Optimizing the output signal-to-noise ratio (SNR) in nonlinear dynamic systems via the "stochastic resonance" (SR) phenomenon has received considerable attention in the past decade [1,2]. More recently, attention has focussed on SR effects in the response of coupled bistable elements interacting via linear [3–5] or nonlinear [6,7] couplings, the latter having potential applicability in the neurosciences. However, the study of SR in terms of important signal processing measures other than SNR has received limited attention [8]. In this paper we consider the signal detection statistics of coupled arrays of nonlinear dynamic elements. We show that these statistics follow the SNR's behavior: they exhibit a maximum as a function of noise, and coupling significantly enhances signal detection over that achieved by a single element.

As a concrete example, we consider a coupled array of overdamped nonlinear dynamic elements, each subject to the *same* external "input," a temporal sine wave of amplitude q in a Gaussian white noise background of one-sided power spectral density D/π (in terms of angular frequency):

$$C_i\dot{u}_i=-\frac{u_i}{R_i}+\sum_{j=1}^{N}J_{ij}\tanh u_j+q\,\sin\omega t+\sqrt{D}\,\xi(t), \qquad (1)$$

where $\xi(t)$ is Gaussian white noise with mean zero and autocovariance $\langle\xi(t)\xi(t+\tau)\rangle=\delta(\tau)$. We designate $u_1(t)$ as the array's "output." Systems of the form (1) have been used to describe connectionist-type electronic neural networks [9]. In such networks, u_i denotes the ith neuron's activation function (membrane potential), and C_i, R_i denote the neuronal input capacitance and transmembrane resistance, with the coupling coefficients (synaptic efficacies) J_{ij} usually determined via a "learning rule." Here, we shall choose the couplings to maximize the output SNR and signal detection statistics [7].

The "reference" element, $i=1$, has bistable dynamics approximated by highly damped motion in a double-well potential. At very low noise levels, the state point oscillates at the bottom of the potential wells for very long times, with very infrequent switching between wells. The output SNR approaches the input SNR because the bottom of each potential well is approximately parabolic, rendering the system's response nearly linear. As the noise level is increased, the response becomes more strongly nonlinear and the output SNR drops. However, as we approach the critical noise level, the time required for the state point to hop between the wells (the Kramers time) approaches half the period of the driving sine wave, rates for switching due to the signal and due to the noise coincide, and the output SNR reverses direction and *rises* to a local maximum (the SR effect). At still higher noise levels, the effect of the potential barrier gradually diminishes, the response becomes less strongly nonlinear, and the output SNR follows the input SNR downward.

To determine whether the SR effect has signal processing applications, we cannot rely on SNR alone. For example, a nonlinear signal processor may output a signal which has infinite SNR but is useless because it has no correlation with the input signal. For signal estimation, relevant measures are mean square error or Bayesian tests [10]. For signal detection, one must consider detection statistics: probability of detection and probability of false alarm. Probability of detection is the probability that the system will report that a signal is present when in fact a signal *is* present. Probability of false alarm is the probability that the system will report that a signal is present when in fact a signal is *not* present. Such statistics are summarized in a plot of detection probability versus false alarm probability known as the receiver operating characteristic (ROC).

As a test case, we choose a well understood signal processing task: detecting a sine wave signal of known frequency and unknown phase in the presence of Gaussian white noise. For this case, one can prove that the optimal detector consists of a linear filter followed by a "decision circuit" [10]. The linear filter measures the power in a narrow frequency band of width $\Delta\omega$ centered on the known signal frequency ω. The decision circuit compares the filter's output to a threshold. If the filter's output exceeds the threshold, then the system's decision is that the sine wave was present. A low threshold leads to high probability of detection and high probability of false alarm, while a high threshold leads to low probability of detection and low probability of false alarm. ROC curves for such a detector are actually

*Electronic address: inchiosa@nosc.mil
†Electronic address: bulsara@nosc.mil

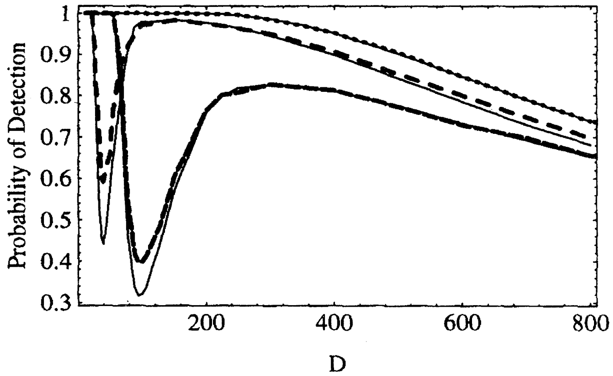

FIG. 1. Probability of detection vs noise strength D, for a fixed probability of false alarm of 0.1. Tightly spaced dashes: output of a single nonlinear element driven by the input signal. Loosely spaced dashes: output of coupled system ($N=2$). Dotted curve: input signal. Solid curves: theoretical prediction. System parameters: $R_i=0.018\,691\,6$, $C_i=1$, $J_{ii}=216$, $J_{12}=50$, $J_{21}=-50$, $A=8$, $\omega=1.225\,22$, and $\Delta\omega=\omega/32$.

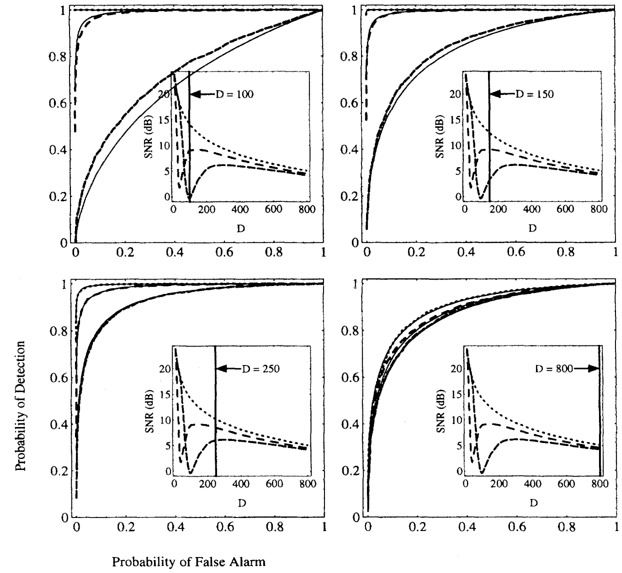

FIG. 2. ROC curves (probability of detection vs probability of false alarm). Tightly spaced dashes: output of a single nonlinear element driven by the input signal. Loosely spaced dashes: output of coupled system ($N=2$). Dotted curve: input signal. Solid curves: theoretical prediction. Insets: SNR vs noise strength D, with the vertical line indicating the value of D used for the ROC's in that panel. System parameters as in Fig. 1.

parametric plots showing the detection and false alarm probabilities as parametric functions of the threshold.

We want to understand how SR in our nonlinear coupled array affects signal detection. Our detector system therefore consists of the coupled array (1) followed by the aforementioned optimal detector.

To measure our system's detection performance using numerical simulation, we begin by computing the system's output SNR and detection statistics via numerical integration of (1) (using the modified Heun method [11]) followed by fast Fourier transform (FFT) of the resulting time series. We define the SNR as the ratio of the signal power to the noise power in the "signal bin" of the FFT, comprising the frequency range $(\omega-\Delta\omega/2,\omega+\Delta\omega/2)$. We estimate the noise power in the signal bin by taking an average of the total power in neighboring bins several bins away (these bins contain noise only). We then subtract this estimated noise power from the total power in the signal bin to obtain the signal power. We use a small time step, $\Delta t=(2\pi/\omega)/8192$, in order to generate noise which has a flat spectrum out to a very high frequency. To avoid aliasing we maintain this sampling rate throughout our computations. To compute the ROC, we repeatedly record the power in the signal bin, both with and without the sinusoidal driving term present, forming power probability density functions (PPDF's) for "signal" or "no signal." Using these statistics, the probability of detection $P_{\rm D}$ for any given threshold power value may be computed by measuring the integral of the "signal" PPDF from the threshold power value to infinity. Similarly, the probability of false alarm $P_{\rm FA}$ equals the same integral of the "no signal" PPDF [10].

Figure 1 shows $P_{\rm D}$ vs noise strength D for a fixed $P_{\rm FA}=0.1$, the threshold being adjusted for each value of D to give the desired $P_{\rm FA}$. Curves are shown for the output of a single driven SR element (tightly spaced dashes), the output of an element coupled with a second element ($N=2$ array, loosely spaced dashes), and the input signal (dotted curve). The solid curves represent approximate theoretical predictions of $P_{\rm D}$ based on the output SNR measured in the numerical simulations [see Eq. (2) below]. The broken curves represent direct measurements of $P_{\rm D}$ from the same numerical simulations. The figure illustrates the "resonance" in signal detection statistics that corresponds to the SR maximum in the SNR. It also shows that using coupled elements brings the performance closer to that of the ideal linear system. Note that a plot of $P_{\rm FA}$ vs D for a fixed $P_{\rm D}$ would look approximately like an inverted Fig. 1.

Figure 2 shows probability of detection vs probability of false alarm. Each of the four panels shows detection statistics for a different noise strength D. The insets show SNR vs D, with the vertical line indicating the value of D used for the ROCs in that panel. As in Fig. 1, solid curves represent approximate theoretical predictions and broken curves represent direct measurements. The figure illustrates a significant enhancement in signal detection for the $N=2$ system (loosely spaced dashes) compared to the single element (tightly spaced dashes). As expected, signal detection is highest for the input signal (dotted curve). If the transducer which picks up the input signal has perfectly linear response, then it is ideal for our task of detecting a sine wave in Gaussian white noise. If the transducer has nonlinear, bistable response, then coupling it into an array can improve signal detection. Indeed, there exist special cases for which the hyperbolic tangent terms in (1) identically cancel, giving the system a perfectly linear response to its input. For example, this occurs for a pair of identical elements ($N=2$, $C_1=C_2$, $R_1=R_2$, $J_{11}=J_{22}$) with couplings $J_{12}=J_{21}=-J_{11}$ and identical initial conditions.

In general, the solution of our system (1) must be found numerically; however, an approximate analytical formulation for the response u_1 of the reference element may be obtained under the following conditions (for details see [7]). The bath elements ($i>1$) must react at a rate much greater than the rate of change of the reference element or the forcing signal. Furthermore, the cross-coupling and forcing terms should not be so large that they overwhelm the basic monostable or bistable dynamics of the elements. This allows us to use Haken's slaving principle [12] to reduce the many-body system to an equivalent one-body system. In the reduced one-body system, the rate at which probability equilibrates at the bottom of the potential wells must be much greater than the rate of the forcing, and the modulation of the potential barrier height must be much less than the noise spectral density, which in turn must be much less than the unmodulated potential barrier height. These conditions allow the use of perturbation theory in deriving the theoretical expressions [1] for the power spectral density of the reference element's motion, from which we obtain the output SNR.

The ROC curves can be predicted from the output SNR. Since we do our signal detection by comparing the power in the signal bin of the FFT to a threshold, and since one bin of the FFT covers a very narrow range of frequencies, the noise spectrum across the bin and in the vicinity of the bin looks approximately constant. Therefore we can approximately model the output of our nonlinear array (which supplies the input to the optimal detector) as a sine wave in white noise with an SNR equal to the array's output SNR, R. For this input, the optimal detector's probability of detection is [10]

$$P_{\rm D}=Q(\sqrt{2R},\sqrt{-2\ \ln P_{\rm FA}}), \tag{2}$$

where

$$Q(\alpha,\beta)\equiv\int_{\beta}^{\infty} z\ \exp\left(-\frac{z^2+\alpha^2}{2}\right)I_0(\alpha z)dz \tag{3}$$

is Marcum's Q function, and I_0 is the modified Bessel function of the first kind and order zero.

This approximating model gives highly accurate results except in a transitional noise range where both intra- and interwell motion contribute significantly to the output SNR. On the SNR vs D plots, this is the regime in the vicinity of the SNR minimum at moderately low noise strength. The response is most strongly nonlinear in this regime. For example, in this regime turning on the sine wave signal (with input noise strength held constant) causes a large increase in both signal *and* noise output power. In this case, calculations based on just the final value of the output SNR (with the sine wave turned on) underestimate the signal detection performance.

To check whether stochastically resonant and coupling-enhanced signal detection relies heavily on the specifics of our SR system, we can consider a quite different system consisting of a chain of overdamped Duffing oscillators [4,8]:

$$\dot{u}_i=ku_i-k'u_i^3+\epsilon(u_{i+1}+u_{i-1}-2u_i)+q\ \sin\omega t+\sqrt{D}\xi_i(t) \tag{4}$$

(with free boundary conditions imposed on the chain). This system (4) differs from the previously considered system (1) in that (a) the noise is local to each element, representing internally generated rather than externally applied noise, (b) the coupling is local, (c) the coupling is linear, and (d) a cubic term is used for the nonlinearity rather than the hyperbolic tangent used in the neuron system. Despite these differences, the output SNR and ROC curves for this system are qualitatively identical to the previous system. Note, however, that the input SNR for this system is infinite because the external input is a noise-free sine wave signal which then gets mixed with internally generated noise. Also, in certain circumstances systems with linear vs nonlinear coupling and global vs local noise do differ qualitatively [7,8].

In summary, we emphasize that (1) the ROC curves mimic the output SNR optimization due to SR, and (2) one can improve the signal detection performance of a stochastic resonator by coupling it into an array of resonators. Away from the SNR minimum, modeling the output of the resonator or array as a sine wave in locally white Gaussian noise permits accurate prediction of signal detection performance.

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